Two Notions of Differential Equivalence on Sboxes

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Abstract. In this work, we discuss two notions of differential equivalence on Sboxes. First, we introduce the notion of *DDT-equivalence* which applies to vectorial Boolean functions that share the same difference distribution table (DDT). Next, we compare this notion, to what we call the γ-*equivalence*, applying to vectorial Boolean functions whose DDTs have the same support. We discuss the relation between these two equivalence notions and provide an algorithm for computing the DDT-equivalence and the γ-equivalence classes for a given function. We study the sizes of these classes for some families of Sboxes. Finally, we prove a result that shows that the rows of the DDT of an APN permutation are pairwise distinct.

1 Introduction

Block ciphers are central primitives in symmetric encryption schemes. Modern block ciphers are designed based on a methodology which guarantees that the cipher is resistant against all classical attacks. The differential cryptanalysis, presented by Biham and Shamir in 1990 [1], is one of the most prominent attacks against block ciphers, and a precise evaluation of its complexity has led to some design criteria on the building blocks in the cipher. The main criterion, which has been introduced by Nyberg and Knudsen [17,18], is the so-called differential uniformity of the Sbox, i.e., of the nonlinear mapping used in the cipher. This parameter should be as small as possible in order to maximize the complexity of differential attacks, and the mappings with the lowest differential uniformity, named APN mappings, have been investigated in many works during the last twenty-five years. Indeed, these mappings are highly relevant for cryptographic applications and they are also optimal combinatorial objects of independent interest. Therefore, this design criterion is at the origin of a whole line of research, including the search for infinite families of permutations with a low differential uniformity, the study of their properties or some classification work (e.g. [5,8,9,10,11,13,17]).

However, besides the differential uniformity of the Sbox, the whole differential spectrum and even the form of the difference distribution table (DDT) are important when the resistance against several variants of differential cryptanalysis is quantified. Obviously, the number of occurrences of the differential uniformity in the DDT of the Sbox corresponds to the number of one-round differentials with the highest probability and should then be minimized. Also, the whole differential spectrum of the Sbox is involved in all known upper-bounds on the maximal expected differential probability over two rounds of an SPN cipher [6,19]. Not only the number, but also the location within the DDT of these maximal values may influence the resistance of the cipher against multiple differential cryptanalysis [3] or truncated differential attacks [14] (see e.g. [2, Section 3.2] for a discussion). When designing block ciphers, it would then be of major interest to be able to start from a desired DDT which guarantees a high resistance against all variants of differential cryptanalysis, and to construct Sboxes having this specific DDT. Instead, the main technique currently available to the designers consists in randomly choosing Sboxes until one with a suitable DDT is found. However, constructing Sboxes from a prescribed DDT is a difficult problem, related to many open issues in the area. The characterization of the valid DDT, i.e. for which there exists at least one function with these particular DDT, is also open.

In the case of APN functions, this general problem corresponds to the problem of determining the *differential equivalence class* of a given function, introduced by Gorodilova [12]. It has also been raised by Carlet in the case of APN functions [7, Pb. 3.11]. It is obviously related to the so-called *Big APN problem*, i.e., the existence of APN permutations operating on an even number of variables. Indeed, it has been long conjectured that bijective APN functions do only exist in odd dimension, until the first ever counter-example over $\mathbb{F}_2^6$ was presented by Dillon et al. [5]. However, the conjecture still stands for any even dimension $n \geq 8$.

Our Contributions. In this work, we provide a new algorithm for computing the differential equivalence class corresponding to a prescribed DDT. We applied this algorithm to find several equivalence classes. Most notably, one of the main problems we focus on is to determine whether the differential equivalence class of a *permutation* over $\mathbb{F}_2^n$ can contain more than 2^{2n} elements. In other words, we wonder whether two permutations F and G with the same DDT necessarily satisfy $G(x) = F(x \oplus c) \oplus d$ for some $c, d \in \mathbb{F}_2^n$. As a result, we found permutations F whose differential equivalence classes contain other elements than the functions $x \mapsto F(x \oplus c) \oplus d$. However, we conjecture that this is only the case when some rows of the corresponding DDT are equals. We also discuss some properties of the DDT of an APN permutation, adding some constraints on the valid DDT for such permutations.

2 Two Notions of Differential Equivalence

Even if the following properties hold in the general case, our work mainly focuses on vectorial Boolean functions with the same number of inputs and outputs, i.e., on functions from $\mathbb{F}_2^n$ into itself. Cryptographic Sboxes are examples of such functions that usually verify additional properties for cryptographic applications, most notably nonlinearity. Although we focus on Sboxes in the remainder of this paper, most of the results can be adapted to general vectorial Boolean functions.

The differential properties of a vectorial Boolean function are related to its derivatives.

Definition 1 (Derivative of a function). *Let F be a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^n$. The* derivative of F with respect to $a \in \mathbb{F}_2^n$ *is the function*

$$\Delta_a F : x \in \mathbb{F}_2^n \mapsto F(x \oplus a) \oplus F(x).$$

The multi-sets corresponding to the images of the derivatives of F are usually represented as a two-dimensional array called the difference distribution table.

Definition 2 (DDT and its characteristics). *Let F be a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^n$. The* difference distribution table (DDT) *of F is the two-dimensional table defined by*

$$\delta_F(a, b) = \#\{x \in \mathbb{F}_2^n : \Delta_a F(x) = b\}, \ \ \forall a, b \in \mathbb{F}_2^n.$$

Two important characteristics of the DDT, introduced in [8, 17] respectively, are as follows:

- *the* differential uniformity of F *is the highest value in the DDT, i.e.*

$$\max_{a,b \in \mathbb{F}_2^n, a \neq 0} \delta_F(a, b).$$

  The lowest possible value for the differential uniformity of a function from $\mathbb{F}_2^n$ into itself is 2 and the functions with differential uniformity 2 are called almost perfect nonlinear (APN).
- *the* indicator of the DDT *is the Boolean function of $2n$ variables defined by*

$$\gamma_F(a, b) = 0 \text{ if and only if } \delta_F(a, b) = 0 \text{ or } a = 0.$$

The previous properties then lead to two different notions of equivalence between Sboxes. We say that

- F and G are DDT-equivalent if they have the same DDT;
- F and G are γ-equivalent if their DDTs have the same support, or equivalently if $\gamma_F = \gamma_G$.

The notion of γ-equivalence has been investigated under the name *differential equivalence* by Gorodilova [12]. It must not be confused with the *differential equivalence* introduced in [20, 21], which refers to another property.

Obviously, DDT-equivalence implies γ-equivalence. However, the converse also holds in some particular cases.

Proposition 1. *Let F and G be two functions from $\mathbb{F}_2^n$ into itself which are γ-equivalent. Assume that, for each derivate of F and G, there exists some integer λ such that the derivative is a λ-to-1 function. Then, F and G are DDT-equivalent. Most notably, this situation holds for quadratic functions or for APN functions.*

Proof. The result comes from the fact that, in this case, the DDT of the function is entirely determined by its support. Assume that, for any $a \in \mathbb{F}_2^n$, $a \neq 0$, $\Delta_a F$ is a λ-to-1 function (where λ may depend on a). Then, the entries of the row in the DDT corresponding to $\Delta_a F$ belong to $\{0, \lambda\}$. Since the sum of all entries within a row equals 2^n, we deduce that λ is a power of 2, and its value can be deduced from the number of elements b such that $\gamma_F(a, b) = 1$ which equals $2^n\lambda^{-1}$. Then, the row corresponding to $\Delta_a F$ is entirely deduced from γ_F. When F is a quadratic function, its derivatives have degree at most 1. Then, $\Delta_a F$ is a 2^d-to-1 function where d is the dimension of the kernel of $\Delta_a F$. □

The previous proposition obviously includes the case of quadratic APN functions studied in [12] and in [22], implying that the γ-equivalent APN functions exhibited in [12] are also DDT-equivalent.

In general, the two notions of differential equivalence do not coincide. The following example exhibits two γ-equivalent functions with different DDTs.

Example 1. Let F and $G : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ be represented by their value tables:

$$F = [\texttt{0x0,0x1,0x2,0x3,0x4,0x5,0x6,0x7,0x8,0x9,0xA,0xB,0xC,0xD,0xF,0xE}],$$
$$G = [\texttt{0x0,0x1,0x3,0x2,0x5,0x4,0x7,0x6,0x8,0x9,0xA,0xB,0xC,0xD,0xE,0xF}].$$

Both DDTs are diagonal with 2×2 blocks, the first block being $\begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$ for both tables. Then, for F, all the diagonal blocks are $\begin{bmatrix} 12 & 4 \\ 4 & 12 \end{bmatrix}$, whereas for G, half of the blocks only are of this shape, the other ones are $\begin{bmatrix} 4 & 12 \\ 12 & 4 \end{bmatrix}$. It is then clear that F and G are γ-equivalent, but are DDT-inequivalent.

In this work, we mainly focus on the sizes of the DDT-equivalence classes and γ-equivalence classes. A lower bound on these sizes is given in the following proposition.

Proposition 2. *Let F be a function from $\mathbb{F}_2^n$ into itself and let ℓ denote the dimension of its linear space, i.e., of the space formed by all $a \in \mathbb{F}_2^n$ such that $\Delta_a F$ is constant. Then, the DDT-equivalence class of F contains the $2^{2n-\ell}$ distinct functions of the form*

$$x \mapsto F(x \oplus c) \oplus d, \quad c, d \in \mathbb{F}_2^n. \tag{1}$$

Proof. The fact that all functions $F_{c,d} : x \mapsto F(x \oplus c) \oplus d$ are DDT-equivalent is well-known (see e.g. [12, Prop. 1]). Now, two pairs (c_1, d_1) and (c_2, d_2) lead to the same function if and only if, for all $x \in \mathbb{F}_2^n$,

$$F(x \oplus c_1) \oplus F(x \oplus c_2) = d_1 \oplus d_2 \ ,$$

which means that $\Delta_{c_1 \oplus c_2} F = d_1 \oplus d_2$, i.e. $(c_1 \oplus c_2)$ is a linear structure and $d_2 = d_1 \oplus \Delta_{c_1 \oplus c_2} F$. Then, the number of distinct functions $F_{c,d}$ equals $2^{2n-\ell}$. □

In the sequel, we consider that two functions are trivially DDT-equivalent if they satisfy the Relation (1) from the above Proposition 2. Moreover, we say that a DDT-equivalent class is *trivial* if its size matches the lower-bound given in Proposition 2.

Another important property of the size of these equivalence classes is the following result proved in [12] for γ-equivalence, which can easily be generalized to DDT-equivalence.

Proposition 3. *Let F and G be two functions which are EA-equivalent, i.e., there exist three affine functions A_0, A_1, A_2 where A_1 and A_2 are bijective such that $G = A_2 \circ F \circ A_1 \oplus A_0$. Then, the DDT-equivalence classes (resp. γ-equivalence classes) of F and of G have the same size. Moreover, the class of G is composed of all $A_2 \circ F' \circ A_1 \oplus A_0$ where F' varies in the class of F.*

It follows that the sizes of these differential-equivalence classes can be computed for one representative in each EA-equivalence class only.

3 Computation of the γ-Equivalence and DDT-Equivalence Classes

We present in this section an algorithm that takes as input a $2^n \times 2^n$ table D filled with nonnegative integers and returns all functions F from $\mathbb{F}_2^n$ into itself, if any, whose difference distribution table has the same indicator (see Definition 2) as the one of D, which we denote γ_D. In other words, our algorithm retrieves the γ-equivalence class of functions of a given table D. Note that one can also derive the DDT-equivalent functions from this class, by post-filtering the functions returned by the algorithm.

Throughout the following sections, we denote binary vectors of $\mathbb{F}_2^n$ by integers and make an extensive use of this notation. The algorithm determines all possible values for $F(i)$, $i = 0, \ldots, 2^n - 1$, by taking into account the constraints imposed by the table D and the values $F(j)$, $j < i$, that have already been computed. It essentially implements a tree-traversal algorithm, where each Level i contains the nodes corresponding to the possible values that $F(i)$ can take. The tree therefore has depth 2^n. There is a natural incentive to implement such algorithms using recursion, which we adopt in the sequel.

From now on, we denote by $\mathcal{R}_i = \{y : D[i][y] \neq 0\}$ the set of column indices of non-zero elements on D's ith row. The algorithm starts running and tries to determine all possible values for $F(0), F(1), \ldots, F(2^n - 1)$. By assuming that all values $F(0)$, $F(1)$, $\ldots$, $F(i-1)$ have already been set, the value $F(i)$ can be computed according to the following relations:

- $F(i) \oplus F(0) = \Delta_i F(0)$ must lie in $\mathcal{R}_i$,
- $F(i) \oplus F(1) = \Delta_{i \oplus 1} F(1)$ must lie in $\mathcal{R}_{i \oplus 1}$,
- $F(i) \oplus F(2) = \Delta_{i \oplus 2} F(2)$ must lie in $\mathcal{R}_{i \oplus 2}$,
- $\ldots$
- $F(i) \oplus F(i-1) = \Delta_{i \oplus (i-1)} F(i-1)$ must lie in $\mathcal{R}_{i \oplus (i-1)}$.

Thus, $F(i)$ should lie in the intersection of the sets

$$\{x \oplus F(0) : x \in \mathcal{R}_i\} \cap \{x \oplus F(1) : x \in \mathcal{R}_{i \oplus 1}\} \cap \cdots \cap \{x \oplus F(i-1) : x \in \mathcal{R}_{i \oplus (i-1)}\}.$$

If this intersection is empty, then the algorithm backtracks and picks another value for $F(i-1)$, from the set of possible values. Otherwise, $F(i)$ is set to the smallest element in the intersection and the algorithm continues by searching for possible values for $F(i+1)$. At this point, it has to be noticed that $F(0)$ can take any given value. However, we explain now a pruning observation that prevents the algorithm from trying all possible 2^n values for $F(0)$ and all possible values for $F(1)$.

Pruning. We can reduce the search space of the algorithm by pruning some branches. The procedure starts, without restriction, by the determination of the images of 0 and 1. We explain now why it is possible to fix those two values and still recover all the other functions for different values of these images.

First, recall that a function $F(x)$ and $F(x) \oplus d$, for any $d \in \mathbb{F}_2^n$, have the same DDT. This implies that there are at least 2^n functions having a certain DDT for any image of 0. We can therefore fix the image of 0 to any particular value, and query the algorithm for functions having this first defined point. All the other functions will then be recovered by translation.

Second, for a defined image of 0, it is not necessary to ask the algorithm to look for every possible image of 1. Indeed, the functions $F(x)$ and $F(x \oplus c) \oplus F(0) \oplus F(c)$, for any $c \in \mathbb{F}_2^n$, have the same DDT. This means that, once $F(0)$ has been fixed, there are as many solutions for any value of $F(1)$ as long as $F(0) \oplus F(1) \in \mathcal{R}_1$. Moreover, remark that there is an even number of functions having the same DDT and the same images in 0 and 1: the functions $F(x)$ and $F(x \oplus 1) \oplus F(0) \oplus F(1)$ are equal in 0 and 1.

One Example. Before giving the pseudo-code of the algorithm, we show a small example of its execution for the $2^3 \times 2^3$ table shown in Figure 1, which corresponds to the DDT of the PRINTCIPHER Sbox [15].

		Δ_{out}							
		0	1	2	3	4	5	6	7
Δ_{in}	0	8	.	.	.	.	.	.	.
	1	.	2	.	2	.	2	.	2
	2	.	.	2	2	.	.	2	2
	3	.	2	2	.	.	2	2	.
	4	.	.	.	.	2	2	2	2
	5	.	2	.	2	2	.	2	.
	6	.	.	2	2	2	2	.	.
	7	.	2	2	.	2	.	.	2

Figure 1: Difference distribution table of dimension $2^3 \times 2^3$ corresponding to the PRINTCIPHER Sbox.

Here are the main steps performed by the algorithm (also see Figure 2):

1. Set $F(0) = 0$
2. Set $F(1) = 1$, as 1 is the minimal value of the set $\mathcal{R}_1 = \{1, 3, 5, 7\}$
3. As $F(2) \oplus F(0) \in \mathcal{R}_2 = \{2, 3, 6, 7\}$ and $F(2) \oplus F(1) \in \mathcal{R}_3 = \{1, 2, 5, 6\}$, $F(2) \in \{2, 3, 6, 7\} \cap \{0, 3, 4, 7\} = \{3, 7\}$. Set $F(2) = 3$.
4. As $F(3) \oplus F(0) \in \mathcal{R}_3 = \{1, 2, 5, 6\}$, $F(3) \oplus F(1) \in \mathcal{R}_2 = \{2, 3, 6, 7\}$ and $F(3) \oplus F(2) \in \mathcal{R}_1 = \{1, 3, 5, 7\}$, $F(3) \in \{1, 2, 5, 6\} \cap \{2, 3, 6, 7\} \cap \{0, 2, 4, 6\} = \{2, 6\}$. Set $F(3) = 2$.
5. As $F(4) \oplus F(0) \in \mathcal{R}_4 = \{4, 5, 6, 7\}$, $F(4) \oplus F(1) \in \mathcal{R}_5 = \{1, 3, 4, 6\}$, $F(4) \oplus F(2) \in \mathcal{R}_6 = \{2, 3, 4, 5\}$ and $F(4) \oplus F(3) \in \mathcal{R}_7 = \{1, 2, 4, 7\}$, $F(4) \in \{4, 5, 6, 7\} \cap \{0, 2, 5, 7\} \cap \{0, 1, 6, 7\} \cap \{0, 3, 5, 6\} = \emptyset$.
6. Go back to Step 4 and set $F(3) = 6$. Compute now any possible values for $F(4)$ by repeating Step 5, with $F(3) = 6$.
7. ...
8. Once $F(7)$ has been fixed, we verify that γ_F is equal to the indicator of D and add it to a list of *solutions*. We then backtrack to find the other solutions.

The two solutions found with the restrictions $F(0) = 0$ and $F(1) = 1$ are $F = (0, 1, 3, 6, 7, 4, 5, 2)$ and $F' = (0, 1, 7, 2, 5, 6, 3, 4)$ as it can be seen in Figure 2. All the γ-equivalent functions can be found by computing $F(x \oplus c) \oplus d$ and $F'(x \oplus c) \oplus d$ for all $c, d \in \mathbb{F}_2^3$. At the end, we obtain 2^6 γ-equivalent functions.

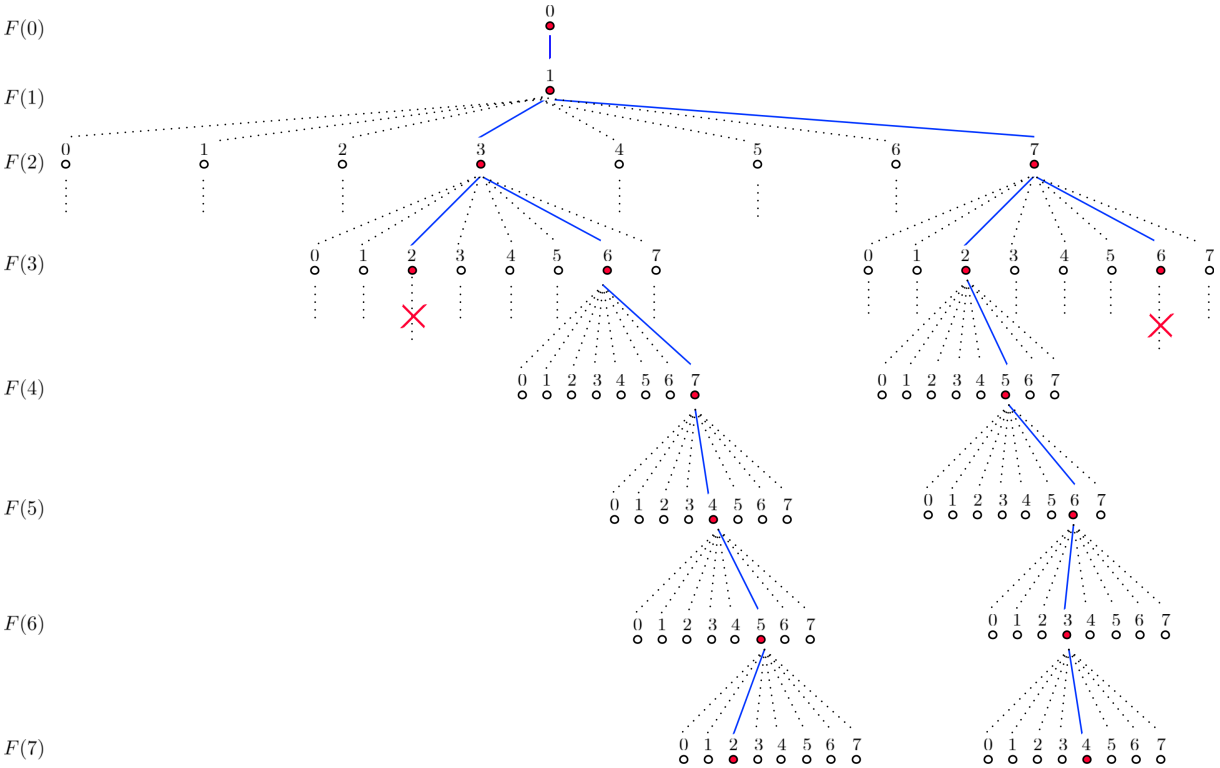

Figure 2: Example of the algorithm's execution on the table of Figure 1.

Algorithm 1 `Main`

Input: A table D of size $2^n \times 2^n$
Output: A list $\mathcal{F}$ of all functions $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ γ-equivalent to the indicator of D

1: $\mathcal{F} \leftarrow \{\emptyset\}$ ▷ Globaly defined
2: $S \leftarrow [0, \min\{\mathcal{R}_1\}, 0, \ldots, 0]$ ▷ $\texttt{len}(S) = 2^n$
3: `RecursifSearch`$(S, 2)$
4: **return** $\mathcal{F}$

Algorithm. In the algorithm, we take the pruning observation into account and only look for functions such that the image of 0 is 0 and the image of 1 is the first possible value. From now on, we denote by S a table of dimension 2^n used to store the intermediate possible images. Then, we denote by F a solution returned by the algorithm, obtained when all the cells of S have been set.

Hence, at the beginning, $S[0]$ is set to 0 and $S[1]$ is set to $\min\{\mathcal{R}_1\}$. The recursive Algorithm 2 is then called for $i = 2$, where i means that the algorithm is searching for candidate values for $S[i]$. It starts by computing the possible values for $S[i]$ on Line 2 and store them in a set $\mathcal{L}$. If this set is not empty, the algorithm tries to compute the next value, $S[i+1]$, for every possible value of $S[i]$. The procedure is repeated until either $S[2^n - 1]$ has been set or $\mathcal{L}$ is empty. In the latter case, the algorithm backtracks to the next possible value in $\mathcal{L}$ at a certain Level i as there was no solution in this branch. In the former case, all the values for S have been set. At this point, we verify (Line 4) whether the function found has the same γ indicator as the table D (resp. it has D as a DDT). Indeed, it is possible that the support of γ_S is strictly included in the one of the indicator of D.

4 Experimental Results

One of the questions we are interested in is the existence of two DDT-equivalent permutations F and G, which are not related by $G(x) = F(x \oplus c) \oplus d$ for some c, d. It is worth noticing that, in the case of non-bijective mappings, such pairs of functions exist. For instance, in [12], $2^{2n+n/2}$

Algorithm 2 `RecursifSearch`

Input: A table S of size 2^n, an integer i

if $i < 2^n$ **then**
  $\mathcal{L} \leftarrow \bigcap_{0 \le k < i} \{x \oplus S[k] : x \in \mathcal{R}_k\}$
else
  if $\gamma_S = \gamma_D$ **then** ▷ Or $\mathrm{DDT}(S) = D$ if we test the DDT-equivalence
    Append S to $\mathcal{F}$
  return
if $\mathcal{L} \neq \emptyset$ **then**
  for all $x \in \mathcal{L}$ **do**
    $S[i] \leftarrow x$
    `RecursifSearch`$(S, i+1)$
else
  return

quadratic functions have been exhibited, which are γ-equivalent (and thus DDT-equivalent) to the Gold function $x^{2^{n/2+1}+1}$ over $\mathbb{F}_{2^n}$ when n is a multiple of 4.

4.1 Results for some Known Functions

Using the algorithm described in the previous section, we have been able to compute the γ-equivalence classes and the DDT-equivalence classes of some cryptographically relevant functions.

It is clear from Proposition 3, that it is sufficient to run the algorithm for a single representative in each EA-equivalence class. Hence, after computation, we can affirm that the size of the DDT-equivalence classes of all APN permutations over $\mathbb{F}_2^n$, with $n \le 6$, is 2^{2n}. This equivalently means that each class is only obtained by translating the input and output of the representative function. More precisely, these representative permutations for each dimension n correspond to:

- for $n = 6$: the so-called Dillon permutation [5],
- for $n = 5$: the five APN permutations described by Brinkmann and Leander in [4, Table 1],
- for $n = 3$: the Gold permutation.

We have also examined all permutations of dimension $n = 4$ with optimal differential uniformity (equal to 4) and nonlinearity from the 16 different affine-equivalence classes given in [16]. The γ-equivalence class for each of them contains exactly 2^8 elements. Since none of these functions has a linear structure, we deduce from Proposition 2 that these 2^8 elements also form their DDT-equivalence class.

Then, none of the permutations with the lowest possible differential uniformity in dimension $n < 6$ has a DDT-equivalence class with size bigger than 2^{2n}. However, it is possible to construct such permutations when we increase the differential uniformity.

4.2 An Example of Non-Trivially DDT-Equivalent Permutations

In this paragraph, we exhibit a permutation F over $\mathbb{F}_2^5$, such that some elements in its DDT-equivalence class are not of the form $F(x \oplus c) \oplus d$ for any $c, d \in \mathbb{F}_2^5$.

We consider the table D, composed by 2×2 blocks of the form $\begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}$ everywhere on the diagonal except for the first block which is $\begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$. By running our algorithm on this table, we recovered 56×2^8 permutations having this DDT. Even by considering the fact that the permutations corresponding to this DDT will necessarily have a linear structure, this number is still higher than the number of distinct functions of the form $F(x \oplus c) \oplus d$, which equals $2^{2\times 5-1}$ as shown in Proposition 2.

In Table 1 are presented two functions F and F' whose DDT is D but for which there is no pair (c, d) such that $F'(x) = F(x \oplus c) \oplus d$ for all x.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F(x)$	0	1	2	3	4	5	6	7	8	9	10	11	13	12	15	14
$F'(x)$	0	1	2	3	4	5	6	7	8	9	10	11	13	12	15	14
x	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$F(x)$	16	17	19	18	20	21	23	22	25	24	26	27	28	29	31	30
$F'(x)$	16	17	19	18	21	20	22	23	24	25	27	26	28	29	31	30

Table 1: Two non-trivially DDT-equivalent permutations.

4.3 A Conjecture on the Size of DDT-Equivalence Classes

We then know two examples of functions having a non-trivial DDT-equivalence class (in the sense that they contain more functions than the ones obtained by translations): some Gold functions studied in [12] and the functions we exhibited above. The common point between these two examples is that they both have non-distinct rows in their DDTs. Besides this property, they seem to have very different characteristics. Indeed, the second ones are permutations with a linear structure, while the first ones are non-bijective and are APN. Moreover, as previously noticed, the APN permutations seem to not have larger DDT-equivalence classes (at least for small dimensions), and the permutations studied in Example 1 have a linear structure but its DDT-equivalence classes are trivial.

These remarks, combined with the computations we have performed using our algorithm, lead us to the statement of the following conjecture.

Conjecture 1. The DDT-equivalence class of a permutation F, such that the rows in its DDT are pairwise distinct, only contains permutations of the form $F(x \oplus c) \oplus d$, with $c, d \in \mathbb{F}_2^n$ (i.e. is trivial).

It is worth noticing that Example 1 shows that the same conjecture does not hold for γ-equivalence.

Hoping to make a step towards the proof of this conjecture, we show in the next section that APN permutations cannot have two equal rows in their DDTs.

5 A Note on the DDTs of APN Permutations

Since the DDTs having at least two equal rows seem to play an important role, a natural question is the following: Is it possible that this situation occurs for some remarkable families of Sboxes? As a partial answer, we prove in this section that all the rows in the DDT of any APN permutation are distinct.

Let F be an APN permutation of $\mathbb{F}_2^n$. We start by stating two simple remarks. The first remark is due to the fact that F is a permutation while the second one is a result of F being APN.

Remark 1. $F(x) \oplus F(y) \neq F(x) \oplus F(z)$, for $x, y, z \in \mathbb{F}_2^n$ pairwise distinct.

Indeed, if we suppose that for some pairwise distinct $x, y, z \in \mathbb{F}_2^n$, we have that $F(x) \oplus F(y) = F(x) \oplus F(z)$, this would imply that $F(y) = F(z)$, which is a contradiction by the fact that F is a permutation.

Remark 2. $\Delta_a F(x) \neq \Delta_a F(y)$, for $x, y, a \in \mathbb{F}_2^n$ with $y \neq \{x, x \oplus a\}$ and $a \neq 0$.

Assuming an equality between the left and the right handsides of the equation, would imply an equality between two images of $\Delta_a F$ not trivially equal, which cannot occur as F is APN.

Theorem 1. *Let F be an APN permutation of $\mathbb{F}_2^n$. Then, the rows of the DDT of F are pairwise distinct.*

Proof. We prove this result by contradiction. Indeed, suppose that the row of the DDT corresponding to the image set of $\Delta_a F$ equals the row corresponding to the image set of $\Delta_b F$, for some $a, b \in \mathbb{F}_2^n \setminus \{0\}$ with $a \neq b$.

The proof then tries to match the values $\Delta_a F(x)$, for $x \in \mathbb{F}_2^n$ with the values $\Delta_b F(x)$, for $x \in \mathbb{F}_2^n$ and to show that this is impossible to do. For this, we show that it is impossible to create a chain of values $x_0, x_1, \ldots, x_{2^n-1}$ such that

$$\begin{aligned} \Delta_a F(x_0) &= \Delta_b F(x_1) \\ \Delta_a F(x_1) &= \Delta_b F(x_2) \\ &\vdots \\ \Delta_a F(x_{2^n-2}) &= \Delta_b F(x_{2^n-1}). \end{aligned}$$

We start by proving the following statement by induction.

Let $x_0, \ldots, x_{k-1} \in \mathbb{F}_2^n$ such that $\Delta_a F(x_i) = \Delta_b F(x_{i+1})$ for all $0 \leq i < k-1$. Then, there are at most $2^n - 4k$ possibilities for choosing x_k such that $\Delta_a F(x_{k-1}) = \Delta_b F(x_k)$. More precisely, x_k does not take any of the $4k$ values $x_i, x_i \oplus a, x_i \oplus b, x_i \oplus a \oplus b$, for $i = 0, \ldots, k-1$.

Basis. Let $k = 1$. Suppose that $\Delta_a F(x_0) = \Delta_b F(x_1)$. Then, the variable x_1 cannot take any of the four values $x_0, x_0 \oplus a, x_0 \oplus b$ and $x_0 \oplus a \oplus b$. Indeed, if we suppose for example that $\Delta_a F(x_0) = \Delta_b F(x_0)$, this translates to $F(x_0) \oplus F(x_0 \oplus a) = F(x_0) \oplus F(x_0 \oplus b)$ which is impossible by Remark 1. We use Remark 1 to prove in the same way the impossibility of the remaining three values. Therefore, there are at most $2^n - 4$ possible values for x_1.

Inductive step. Suppose that for all $i < k$ there are at most $2^n - 4i$ possibilities for choosing x_i and that x_i cannot take any of the values in the set $\{x_j, x_j \oplus a, x_j \oplus b, x_j \oplus a \oplus b | 0 \leq j < i\}$. We show in the following that there are at most $2^n - 4k$ possibilities for choosing x_k.

We have that $\Delta_a(x_{k-1}) = \Delta_b(x_k)$. By Remark 1, we get that $x_k \neq \{x_{k-1}, x_{k-1} \oplus a, x_{k-1} \oplus b, x_{k-1} \oplus a \oplus b\}$. We show now that $x_k \notin \{x_i, x_i \oplus a, x_i \oplus b, x_i \oplus a \oplus b \mid 0 \leq i \leq k-2\}$. Indeed, suppose for example that $x_k = x_i$ for some $0 \leq i \leq k-2$. We have that

$$\begin{aligned} \Delta_a F(x_{i-1}) &= \Delta_b F(x_i), \\ \Delta_a F(x_{k-1}) &= \Delta_b F(x_i). \end{aligned}$$

By adding these equations, we get that $\Delta_a F(x_{i-1}) = \Delta_a F(x_{k-1})$. By the induction hypothesis, $x_{k-1} \neq x_{i-1}$ and since F is APN we get a contradiction by Remark 2. The other contradictions are obtained in a similar way by the induction hypothesis and Remark 2.

We show now that it is impossible to construct such a sequence $x_0, \ldots, x_{2^n-1}$. Indeed, we can see, that for choosing for example a value for x_k for $k = 2^{n-2}$, there are $2^n - 4 \cdot 2^{n-2} = 0$ choices left. Therefore, we conclude that if F is an APN permutation of $\mathbb{F}_2^n$ all rows of the DDT must be pairwise distinct. □

6 Conclusion

In this paper, we investigated two different notions of differential equivalence, the DDT-equivalence and the γ-equivalence, and provided an algorithm to compute both equivalence classes for a given vectorial Boolean function. During our experiments, we encountered permutations over $\mathbb{F}_2^n$ whose differential equivalence class contains more than 2^{2n} elements. We conjectured in this paper that functions having a non-trivial DDT-equivalence class may relate to the number of distinct rows in their DDT. Finally, an interesting future direction would be to study the differential equivalence classes, and in particular the sizes, of functions either in higher dimensions and/or without any particular structure.

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